Multirate Sampling, Generalized Alias, and Signal

Rectification as a Source of DC Offset Errors

in Magnetometer Experiments

Charles P. Sonett

Space Sciences Division
Ames Research Center, NASA, Moffett Field, California

(NASA-TM-X-57305) MULTIRATE SAMPLING, GENERALIZED ALIAS, AND SIGNAL RECTIFICATION AS A SOURCE OF DC OFFSET ERRORS IN MAGNETOMETER EXPERIMENTS (NASA) 32 p

N76-70901

Unclas
00/98 29443

Abstract

It is customary to time share or commutate the return data link from spacecraft. This means that the data transmission is periodically interrupted. Because of the dual rate, a beat phenomenon appears between the two periods which results in the generation of a double infinity of switching tones. Sampling of time series data under these conditions results in violation of the fast rate Nyquist limit with the consequent folding of the spectral density distribution function of the data to both higher and lower frequencies unless the presample filter bandwidth is determined by the commutation frequency. In the time domain, the consequence is to introduce rectified components of the spectrum at zero frequency. This is the companion folding to that occurring in spin modulation especially in a turbulent or highly radiative hydromagnetic medium.

These conditions are applied to a hypothetical magnetometer experiment and it is shown that, depending upon the frequency content, the data can be intrinsically aliased at the commutation switching rate of the telemetry system and, further, second order aliased in the data reduction sampling procedure. The presence of the doubly aliased time series for each of the magnetic field components can lead to potentially serious errors depending upon the form of the spectral amplitude functions which, in turn, are unspecifiable if folded.

1. INTRODUCTION

It is common practice to time share telemetry among spacecraft experiments; that is, each experiment in turn occupies the entire transmission link for a fraction of the total time available. In this process, called multirate sampling, the instantaneous sampling rate is likely to be high compared to the rate of commutation. The importance of correctly understanding the multirate problem arises from the requirements of the Nyquist criterion which states that a function band limited to the frequency domain $\omega \leq \omega_0$, where ω_0 is some cut-off, is completely specified by periodic sampling at intervals of π/ω_0, except for harmonic terms having zeros at the sampling points [Bracewell, 1965]. Increasing the sampling interval time causes the spectral orders to overlap so that higher frequencies, defined by ω_0, appear within the passband.[1] This is called alias and for periodic sampling, which does not observe the Nyquist criterion, results in the frequency shifting of the spectral terms.

In the singly periodic case, folding to dc (zero frequency) rarely is experienced except for extreme violations of the sampling theorem. However, for multirate sampling, it is easy to show that folding to dc can occur commonly. In general, spectral folding under the conditions of multirate sampling is characterized by both upward and downward frequency folding for systems which are low pass filtered. When folding extends to zero frequency the effect is demodulation; nonzero spectral components are shifted to the zero frequency axis.

In an earlier paper [Sonett, 1965] the multirate problem was explored in a general way including when the natural spectrum of frequencies in the data were contained in the neighborhood of the spin

---

[1]Strictly speaking, upward folding in the multirate case can also take place as is seen by considering the case discussed later.

rate of the satellite. From the standpoint of folding both cause essentially the same effect when spectral components are folded to zero frequency, i.e., vector additions to the dc value of the data are introduced.

2. PRELIMINARIES

The process of sampling can be coded in a number of ways. It is even possible to construct a sampling sequence which is multiply periodic and recover data where the Nyquist limit corresponds to the average sampling interval. To do this requires that very special presample filters be applied to the data. Bracewell [1965] points out that in the limit where pronounced multiple periods are observed, sampling can be done by taking the function value together with n derivatives at widely separated points [cf. Linden, 1959]. However, this procedure has no apparent advantages in the design of a sampling switch system, and there are disadvantages in the presence of noise which may make such sampling unattractive. It is also fundamental to most sampling theories to assume a stationary spectrum. This is an important limitation when attempting to expand bandwidth by the use of pseudonoise codes for the addressing of a switching matrix. For a birate switch where two δ function switch sequences are interlaced, Bracewell [1965] shows how data can be recovered at the mean switching rate in accordance with the Nyquist criterion, provided the presample filter has a weighting function of the general form

$$\frac{\sin 2\pi t}{2\pi t} - [\pi \cot(\alpha\pi)]\, t \left(\frac{\sin \pi t}{\pi t}\right)^2 .$$

where α is the sample pulse interval.

To compute specific alias errors from data, the phase content must be preserved so that the form of the spectral function cannot be stated in terms of the power but must be given as a spectral density distribution in complex amplitude. If the Fourier coefficients are completely specified, then it is possible to compute the properties of the time function. We shall be especially concerned with the time function in the neighborhood of zero frequency since that is the measure of static field components. The basic limitations in establishing a meaningful measure of errors are due to the irreversibility of the sampling which masks the form of the function under conditions where the data are aliased. Thus we can only estimate some model errors and show how they can grow to significant proportions. <u>Finally, this leads to the conclusion that unless specific assumptions are made regarding the spectral content of the data, it is meaningless to assign an experimental accuracy</u>. We have chosen to consider only a scalar time series for discussion even though magnetic fields are vectors. Certain other effects can creep in when the vector field is reconstituted from the component measurements. However, we believe it is reasonable to assume that the situation cannot grow simpler. It is further assumed that system linearity is preserved throughout the data chain so that one can, with assurance, move reversibly from the time to the transform domain.

The theoretical treatment must also be based upon the assumption that the spectrum of the data to be examined is time stationary. Otherwise, there is no possibility of analysis with the exception of slowly varying data to which trending theory applies. We make the assumption that sufficient data are available so that processing takes place over a time span sufficient to define the spectrum. It is also assumed that the most general representation of the function for the time series contains both even and odd parts, though we shall deviate from this for special purposes of illustration.

3. THE SPECTRAL DISTRIBUTION FUNCTION FOR COMBINED δ FUNCTION APERTURE SAMPLING

We discuss here the spectral distribution function for the case of combined δ function-aperture sampling. The discussion of the properties of the function which describes the distribution of tones is then followed by application to a hypothetical case, and the rectification offsets which are possible are explored in detail.

Given a periodic sequence of δ functions,

$$g(t) = \sum_{-\infty}^{\infty} \delta(t - nT_1) \tag{1}$$

it is well known that convolution of the transform of $g(t)$ with the transform $F(\omega)$, of a time series, $f(t)$, yields the spectral replication [Bracewell, 1965].

$$F(\omega) = \frac{1}{T_1} \sum_{n=\infty}^{\infty} F(\omega - n\omega_s) \tag{2}$$

where $F(\omega)$ is the ensemble of spectral density distributions summed over all orders of replicated spectra, $\omega_s = 2\pi/T_1$, and T_1 is the time interval between the sampling 'on times.'

When the function $g(t)$ is modulated or interrupted by a function $g(t)^*$ which blanks the sampling pulses in the manner shown in Figure 1, Figure 1
$F(\omega)$ is generalized to [Sonett, 1965]

$$\sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \frac{1}{T_1}\left(\frac{\tau}{T_2}\right)\frac{\sin \pi k\tau/T_2}{\pi k\tau/T_2} F(\omega - n\omega_s - k\omega_\tau) \qquad (3)$$

It is observed that, in general, the function F is a complex quantity for any value of the argument. The complex nature of F derives, in turn, from $f(t)$ which quite generally contains both even and odd parts. The expression in equation 3 carries the amplitudes and the original order of replicated spectra; in addition, it contains a satellite structure of spectra centered about each of the original members of the sequence generated for the periodic case. The fast sampling period T_1 corresponds to the sampling rate given in equation 2; T_2 is the period of the modulation, $g(t)^*$; n is the order of the primary spectra as given in equation 2; k is the satellite multiplet order; τ the 'on time' for $g(t)^*$; $\omega_s = 2\pi/T_1$; $\omega_\tau = 2\pi/T_2$; and $F(\omega - n\omega_s - k\omega_\tau)$ represents replication of the original data spectrum at order n,k. There exists a double infinity of spectra corresponding to the extension of the single infinity given in equation 2. The factor preceding F is the amplitude modulation to be applied to a particular order k. The highest amplitude spectra obtained are governed by the factor $1/T_1(\tau/T_2)$. Thus, as the duty cycle of the commutation (τ/T_2) is changed, the amplitude of the maximum $(k = 0)$ spectra various accordingly. The number n denotes the spectral order of the terms given for simple periodic sampling (equation 2), and k denotes the satellite spectral order for each n. The amplitudes of the spectra are, therefore, modulated according to the order number, k.

The generation of the function, whose spectrum $F(\omega)$ is displayed in equation 3, is illustrated in Figure 1 which shows the three functions forming the total time series. Figure 2 shows the tones formed from the switching functions alone. The generation of the subsidiary tones can be regarded as due to a beat phenomenon between the fast and slow rates in all order of spectra. The last step in the construction consists in using the switching functions for sampling a real time series. Then replications of the original data spectrum are generated about each of the lines of the switching tones. Equation 3 corresponds to an extension to higher order of the usual concept of spectral folding or alias. The important distinction is that for the extended case, the relevant Nyquist limit required to guarantee no spectral folding is determined by T_2 rather than T_1. The character of the folding is determined by the duty cycle, τ/T_2, and the spectral distribution function, $F(\omega)$. Equation 3 shows that secondary lobes are present as determined by the modulation envelope.

Figure 2

In order to express the meaning of equation 3 in a physical manner, consider the following restricted problem. Given a spectral distribution, then $F(\omega)$, as before, is band limited to satisfy the Nyquist limit imposed by T_1. The band extends from zero to one-half the frequency of the dominant line, $n = 1$, which is recognized as the first-order switch tone for the simple periodic sampling with period T_1. We now restrict consideration to the first lobe of the modulation function $\sin x/x$ where $x = \pi\tau/T_2$. As an example we shall take a duty cycle with commensurate periods where $\tau/T_2 \cong 0.25$ which gives zeros of $\sin x/x$ for $k = 4, 8, \ldots.$ Thus these spectral lines are suppressed. (For particular values of $\tau\omega$, the vectors are equally dense in phase and the spectrum is suppressed. For this reason, certain switching tones do not occur in the generation of the total switching spectrum.)

The spectral lines examined here and upon which the data spectral function, $F(\omega)$, is replicated are given by $n = 0, -3 \leq k \leq 3$ <u>for the zeroth order of n</u>. There will be some contribution from spectra associated with $n = 1$. A little reflection shows that these contributions are associated with $n = 1, k = -3, -2, -1$. There are consequently a grand total of ten overlapping spectra counting only first lobe contributions. Those in the negative frequency domain are folded about zero and therefore are degenerate; the total measured count becomes seven. The complete spectral density which is measured is due to the composite values arrived at from summing these members. The various spectra are shown in Figure 3 in exploded form for clarity. Figure 3
All members have an equal span in frequency; however, the amplitudes are governed by the modulation, $\sin x/x$. The total spectral intensity is arrived at from graphically adding the contributions from the various spectra or alternatively summing equation 3. For the case cited, $\tau/T_2 \cong 0.25$, the spectral density amplitude distribution is given by

$$\overline{F(\omega)} = \frac{1}{4T_1}\Bigg[F(\omega) + \frac{2\sqrt{2}}{\pi} F_{0,1}(\omega - \omega_T) + \frac{2}{\pi} F_{0,2}(\omega - 2\omega_T)$$

$$+ \frac{2\sqrt{2}}{3\pi} F_{0,3}(\omega - 3\omega_T) + \frac{2\sqrt{2}}{3\pi} F_{1,3}(\omega - \omega_S + 3\omega_T)$$

$$+ \frac{2}{\pi} F_{1,2}(\omega - \omega_S + \frac{2\sqrt{2}}{\pi} F_{1,1}(\omega - \omega_S + \omega_T)\Bigg] \qquad (4)$$

where all but the first term are alias contributions.[2] The remarkable form of equation 3 and the diagram of Figure 3 are strongly reminiscent of diffraction theory. To workers in that field, it is immediately evident that the sampling procedure being discussed here has a transform which is identical to the space pattern of diffraction established through a slit whose aperture corresponds to the gate function, $g^*(t)$, followed by a uniform grating.

It is clear from inspection of equation 3 how increasing the sampling interval can decrease the alias. Consider the case where data are sampled at a high rate for an extended period of time but with preservation of the duty cycle cited. Since T_2 is long, the subsidiary spectra will converge and the first zero of $\sin x/x$ will be achieved quickly. If the true data spectrum has important high frequency components, the data bandwidth will tend to be large compared to the width of the first lobe of the modulation function, $\sin x/x$. Therefore, the fractional error in the proper position of an interval from a secondary spectrum will decrease, and the spectrum will appear to approach strict superposition. In short, the switch tones from T_1 will tend to coalesce about the central terms $n = 1$, $k = 0$. This is the basis of

[2]Equation 4 tacitly ignores phase for the sake of simplicity and assumes all contributions are purely real (see section 5).

the intuitive conclusion that a commutated time series with very long 'on time' can effectively generate a useful spectrum without necessarily having serious folding problems. An example would be a data link that samples for one day every other day.

The generalization of the usual alias problem expressed for periodic sampling is commonplace because of the use of commutation or time sharing in satellite data transmission. Without periodic sampling and subsequent buffering prior to time sharing, the alias problem is greatly complicated and cannot be satisfactorily resolved if the Nyquist limit for the lower frequency, $g(t)^*$, has not been observed in the presample filtering operation.

4. ARITHMETIC AVERAGING OF THE FAST SAMPLES

Examination of the form of the spectral distribution function discloses that reduction of the order of the alias might be entertained by a procedure which can be described simply as burst averaging the 'on-time' data. This corresponds to convolving the data with a rectangular weighting function, equivalent to numerical filtering with a $\sin x/x$ transfer function. However, the process does not produce a running average and therefore the bandwidth, albeit aliased, is determined by the interaveraging time, T_2. We can represent the operation as follows. The interrupted function $g(t)$ is constituted of bursts of deltas, each having j members; therefore,

$$g(t)g(t)^* = \sum_{n=-\infty}^{\infty} \sum_{q=1}^{j} \delta(t - q\tau - nT_1) \qquad (5)$$

where τ is the interval between the fast occurring pulses, j is the order number in the subsequence of fast pulses, n is the order of the pulse group counted over a time interval, T_2, where T_2 is the period associated with the repetition of a pulse subsequence. We reorder equation 5 by assembling all pulses having the same index, j. Therefore,

$$g(t)g(t)^* = \sum_{q=o}^{j} \sum_{n=-\infty}^{\infty} \delta(t - q\tau - nT_1) \qquad (6)$$

The reordered sequence is to be understood as a finite set of sequences, each of which has a strictly periodic time of T_2. The procedure for reordering is shown diagrammed in Figure 4. From the shifting theorem, the companion transforms in the frequency domain are represented by

Figure 4

$$\overline{G(\omega)} = \frac{1}{T_1} \sum_{q=o}^{j} \sum_{n=-\infty}^{\infty} G(\omega - n\omega_s)e^{iq\tau\omega} \qquad (7)$$

It is apparent that one can operate either in the time or frequency domain in order to shift the spectra into strict superposition. In the time domain, the sample pulses in the sampling interval when $g(t)^* = 1$ are inversion delayed with the first sample delayed by $j\tau$. The last pulse is given zero delay and, in general, the delay will be given by $q\Delta$. The delay must be denumerable, and in the process outlined, the processing is a posteriori. In the frequency domain, the equivalent operation as indicated from equation 7 is to apply a linear phase shift with frequency of the amount $iq\tau\omega$.

The previous discussion is the mathematical description of what can nearly be made to be simple arithmetic averaging. From the spectral standpoint, however, it is well to point out that the Fourier coefficients for each of the samples are complex and that it is the process of time or phase shifting which rotates these into the real axis so that simple addition can be performed.

Consideration of the process just described indicates a flaw in its application since it has not been stated that the presampled data has been properly Nyquist limited. To do this requires that the high frequencies be removed. When this is done and only those frequencies remain which properly observe the Nyquist limit for the low frequency switching, the averaging process can be said to have been carried out in an intrinsic manner by the analog filter. In short, averaging or what is equivalent, the application of a weighting function, can only be carried out in the instance considered prior to the sampling operation. When done after sampling at the fast rate the high frequency alias is permanently imbedded in the data.

The fallacy in this approach when the proper Nyquist frequency is observed lies in the lavish and nonoptimum use of bandwidth, i.e., there is no point in sampling faster than at a rate allowed by the Nyquist limit.[3]

5. PHASE

Phase information is not carried explicitly in equation 3. To compute folding errors it is necessary to view $F(\omega)$ as a complex vector

[3]An exception to this would require utilization of the complicated weighting function described earlier with its potentially disadvantageous characteristics.

or else initially restrict computation to rms sums of spectral strips which contribute to some spectral region, say $\omega \cong 0$.

To examine phase information, the time function $f(t)$ is separated so that

$$f(t) = f_e(t) + f_o(t)$$

where $f_e(t) = f_e(-t)$ and $f_o(t) = -f_o(-t)$ being, respectively, the even and odd parts. We take $f(t)$ to be real and thus the Fourier transform is Hermitian; i.e., the real part is even and the imaginary part odd. Thus if the transform of $f(t)$ is $G(\omega) = \chi(\omega) + i\varphi(\omega)$

$$\varphi(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_o(t) e^{-i\omega t}\, dt \tag{8}$$

and

$$\varphi(\omega) \Big|_{\omega \to 0} \longrightarrow 0$$

The transform of the sampled function (equation 3) is expressed as

$$\sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \left[\xi_{nk}(\omega, \omega_s, \omega_T) + i\eta_{nk}(\omega, \omega_s, \omega_T) \right] \tag{9}$$

and it follows that the terms sought about the origin are given by

$$\alpha = \lim_{\omega \to 0} \frac{\Sigma\, \Sigma \xi_{nk}(\omega)}{\Delta\omega} \Bigg|_{\omega=0} \quad ; \qquad \beta = \lim_{\omega \to 0} \frac{\Sigma\, \Sigma \eta_{nk}(\omega)}{\Delta\omega} \Bigg|_{\omega=0} \tag{10}$$

It follows further from the above arguments using the properties of the shifting theorem that the phase at the origin is given by

$$\tan^{-1} \frac{\beta}{\alpha} = \tan^{-1} \frac{\eta_{00}}{\xi_{00}} + \tan^{-1} \frac{\eta_{01}}{\xi_{01}} + \ldots \tag{11}$$

The requirement that all aliases disappear is given by

$$\left(\sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \xi_{nk}\right) - \xi_{oo} = \left(\sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \eta_{nk}\right) - \eta_{oo} = 0 \qquad (12)$$

Each of the spectral replications generated carry the same phase as the original function, $F(\omega)$. To the phase functions must be added the shift $e^{iq\omega\tau}$ according to equation 7. Thus the phase function for a particular replication is given by

$$\varphi_{ij}(\omega) + q\omega\tau$$

where

$$\varphi_{ij}(\omega) = \tan^{-1}\left(\frac{\eta_{ij}}{\xi_{ij}}\right)$$

and $q\omega\tau$ is the contribution from sample lag time. At $\omega = 0$, $q\omega\tau \to 0$ for any spectral replication. However, without the complete complex form for $F(\omega)$, one cannot hope to resurrect the correct dc field values when the folding overlaps $\omega = 0$.

Rectification and demodulation error estimates. The process whereby a nonzero spectral contribution is shifted down to zero frequency is exactly equivalent to the operation of demodulation or rectification and is tantamount to the conventional operation of demodulation of communication practice. Any folding operation which carries $\omega \to 0$ is a contributor to the rectification process whether it arises from sampling or from another effect, such as spin modulation folding. Thus, in general, it is possible to envision several means of making contributions which are assembled in the neighborhood of the origin and which have arisen from different portions of the original spectrum. Using equation 4, the absolute errors both for the real and imaginary parts of the spectrum in the neighborhood of the origin and due to the folded contributions are

$$\overline{\xi(\omega)}\Big|_{\omega=0} = \frac{1}{4\pi T_1}\Big[2\sqrt{2}\ \xi_{0,1}(\omega - \omega_T)$$

$$+ 2\xi_{0,2}(\omega - 2\omega_T) + \frac{2\sqrt{2}}{3}\ \xi_{0,3}(\omega - 3\omega_T)\Big]\Big|_{\omega=0} \qquad (13a)$$

$$\overline{\eta(\omega)}\Big|_{\omega=0} = \frac{1}{4\pi T_1}\Big[2\sqrt{2}\ \eta_{0,1}(\omega - \omega_T)$$

$$+ 2\eta_{0,2}(\omega - 2\omega_T) + \frac{2\sqrt{2}}{3}\ \xi_{0,3}(\omega - 3\omega_T)\Big]\Big|_{\omega=0} \qquad (13b)$$

Assume now for computational purposes a linear ramp of the form

$$\left.\begin{array}{l} \xi(\omega) = a - b\omega\ ; \qquad \xi(\omega) = 0 \quad \text{when } |\omega| < \omega_c \\ \qquad \eta(\omega) = 0 \quad \text{everywhere} \end{array}\right\} \qquad (14)$$

where a is taken as unity and $b = 1$ for $\omega_1 = (\pi/T_1)$. The tacit assumption is that $F(\omega)$ is real everywhere making $f(t)$ Hermitian. Other possibilities exist and are perhaps more realistic. However, for our illustrative case, simplicity dictates the choice made. This lies somewhere between a maximum and minimum possible set of errors. The worst case would be when either ξ or η were zero everywhere and the spectral contributions were summed arithmetically. A minimum exists when the conditions given by equation 12 are satisfied and all folded contributions sum to zero.

In our example we attempt to strike a mean by making all Fourier components real but to rms the sum.

Table 1 is a compilation of errors for the linear ramp and rms summing with variable bandwidth spectra characterized by values normalized to $\bar{\omega} = 1$ for $\omega = \pi/T_1$, the original singly periodic Nyquist value. It is seen that the errors grow rapidly as the bandwidth of the data spectrum is increased as would be expected upon heuristic grounds. Also, the assumption of errors smaller than about 10% means that the spectrum should be assumed to become insignificant beyond one-half the commutation frequency. The calculation assumes zero contribution from the minor lobes of $n = 1$ and from higher order terms, i.e., $n = 2 \ldots$. That this underestimates our simple computation is seen from the divergence which would exist if an essentially flat spectrum were taken. In that case, the limiting parameters would possibly be the true quadrature component of the spectrum together with the finite width of the delta function sampler which would cause the contributions to be cut off eventually.

The assumption that $n(\omega) = 0$ everywhere is equivalent to $f_0(t) = 0$ for all t. The computation invokes the phase shift, $q\omega\tau$, indirectly by the rms procedure. This seems reasonable; a more exact computation must question the assumption that $F(\omega) = 0$. Cases will arise where the phases of all folded contributions cancel at $\omega = 0$. More specifically, large errors that cannot be ignored are predicted for the heavily aliased situations as our sample calculation of rms error illustrates.

Table 1

6. DISCUSSION

Analysis of the multirate problem progresses most easily in the frequency domain; it is more difficult to give a comfortable physical picture of the way in which the folding of spectral components can cause errors in the final values of the parameters sought. The phenomenon of multirate folding, when transformed to the time domain, becomes one of the loss of certain components in the spectrum. Besides the earlier observations, this means that beat notes between tones can be hidden. The analogy is complicated for we deal with a spectral continuum; therefore, beating or mixing of tones becomes a very complicated process. Nevertheless, since for the example chosen the duty cycle is 1/4, it is reasonable to expect that mixing could be subjectively lost. This means, in turn, that amplitudes of the 'one time' data although carrying instantaneous information, provide erroneous summaries over longer times, since either nodes or maxima could be sampled and there is no way to distinguish the two cases. The complication of discussing the higher order alias problem in the time domain is brought out in the text and no more can be done but to give an intuitive guide for the results of the spectral folding as they will take place in the time series which is returned by the spacecraft. For quantitative examination, we must return to the spectral domain according to the earlier arguments. It is clear that from the most fundamental standpoint it is impossible not to have the spectral folding issue carry over into the time domain through the process of inverse Fourier transformation. Lastly it becomes clear how any high frequencies present in the time series could be the result

5

of, or modified by a beat or resonance phenomenon according to the folding of the spectra. Any attempt to circumvent the folding restriction violates the fundamental theorem of sampling. In short, measures of the mean field and/or of the variance in the field are subject to the risk that long period waves having a broad span of frequencies can beat together to produce artificially high or low values of the measured components, depending upon the particular character of the spectra.

An additional comment sometimes made is that instantaneous field values are more suitable for geophysical study than when filtered to satisfy the Nyquist criterion. Fast sampled short data burst which exceed the bandwidth allowed by the sample period have little physical relevance unless the spectrum can be shown to be free of alias-prone components. In that case there is no point in "super-Nyquist" sampling.

The point is sometimes raised that difficulties which might occur in the spectral domain are not necessarily reflected in the time domain where, for the case cited, the data have the form of an interrupted time series. We believe that this objection stands corrected. Although spectral information as usually presented lacks phase since the moduli of the complex Fourier coefficients alone are given (as in power spectra), this representation nevertheless provides a powerful means of understanding geophysical data. For example, it would be difficult to determine the Q of a resonance from time series alone. Another example lies in the cascading of eddys in the decay of turbulence. It would take considerable endeavor to attempt a meaningful formulation of this problem with recourse only to the time domain.

It is clear that the optimum bandwidth for sampled data occurs most simply for periodic sampling. Therefore, in commutation all experiment lines should be evenly interlaced so that burst transmission is not used. The alternative, sometimes employed, is to sample at a periodic rate and buffer data so that they are read out compatibly with the commutated telemeter. However, this is expensive in circuit complexity and reliability. Also it is not always possible to make the sampling and buffering commensurate with the commutation rate, and therefore, some decrease of bandwidth from the optimum may be necessary so as to observe the Nyquist limit.

Acknowledgment. I wish to thank D. S. Colburn for many helpful discussions of the problem of sampling and J. M. Modisette for drawing my attention to the coalescing and degeneracy introduced for long sample 'on times.'

REFERENCES

Bracewell, R., The Fourier Transform and Its Applications, McGraw-Hill, 1965.

Sonett, C. P., Modulation and sampling of hydromagnetic radiation, Space Res., 6, Proc. Sixth International Space Sciences Symposium, COSPAR, May 1965, Mar del Plata, Argentina (also NASA TN D-2950, 1965).

Linden, P. A., A discussion of sampling theorems, Proc. IRE, 47, 1219 (1959).

TABLE 1

Normalized ramp cutoff, $\omega_0 = \pi/T_1$	rms folding error, %
0.25	0
0.50	9
0.75	18
1.00	25
2.00	37
4.00	>200

Fractional error in the neighborhood of $\omega = 0$ due to folded contributions. The frequency is normalized to $\bar{\omega}^* = 1$ at π/T.

*This value is computed solely from terms where $n = 0$. The correct value would include contributions from $n = 1$ and the error would be higher.

FIGURE TITLES

Fig. 1.- The three functions which form the complete sampled time series. The original data are given by $f(t)$, the initial time series, $g(t)$ is the fast sampling sequence of delta functions, and $g^*(t)$ is the modulation or commutation which switches $g(t)$.

Fig. 2.- The spectrum of switching tones generated by $g(t)$ and $g^*(t)$. The primary order, n, corresponds to the high sampling rate of $g(t)$ while the subsidiary spectra are identified by the order k. The case shown is for a duty cycle of 0.25 which has zeros for $k = 4, 8, 12, \ldots$. This case is discussed at length in the text. Each of the switch tones forms the axis for a replication of the data spectrum. These overlapped spectra are not shown here. The original Nyquist limit given by the fast rate sampling corresponds to band limiting at the first zero, that is, $n = 0$, $k = 4$.

Fig. 3.- The spectrum of switching tones generated by $g(t)$ and $g^*(t)$. The case shown is for $n = 0$ and a duty cycle of 0.25. The subsidiary spectra, $k = 1, \ldots$, are shown up to the first zero, $k = 4$ and do not include the negative ($n = 0$, $k = -1, -2, -3$) folded terms. The original Nyquist limit given by the fast rate sampling corresponds to band limiting at the first zero, that is, $n = 0$, $k = 4$. The indicated amplitudes at $\omega = 0$ are to be multiplied by $F(\omega - \omega_S \pm n\omega_T)$ according to equation 4 to obtain the contribution of each folded spectrum at $\omega = 0$.

Fig. 4.- A simplified view of the reordering of the sampling delta functions for an interrupted $g(t)$, that is, with $g^*(t)$ modulation. Here we ignore the modulation in the sense discussed earlier and merely reorder the interrupted sequence into periodic sets of period T_2. This allows the process of secondary spectral generation to be displayed from the standpoint of complex Fourier coefficients as shown in the text.

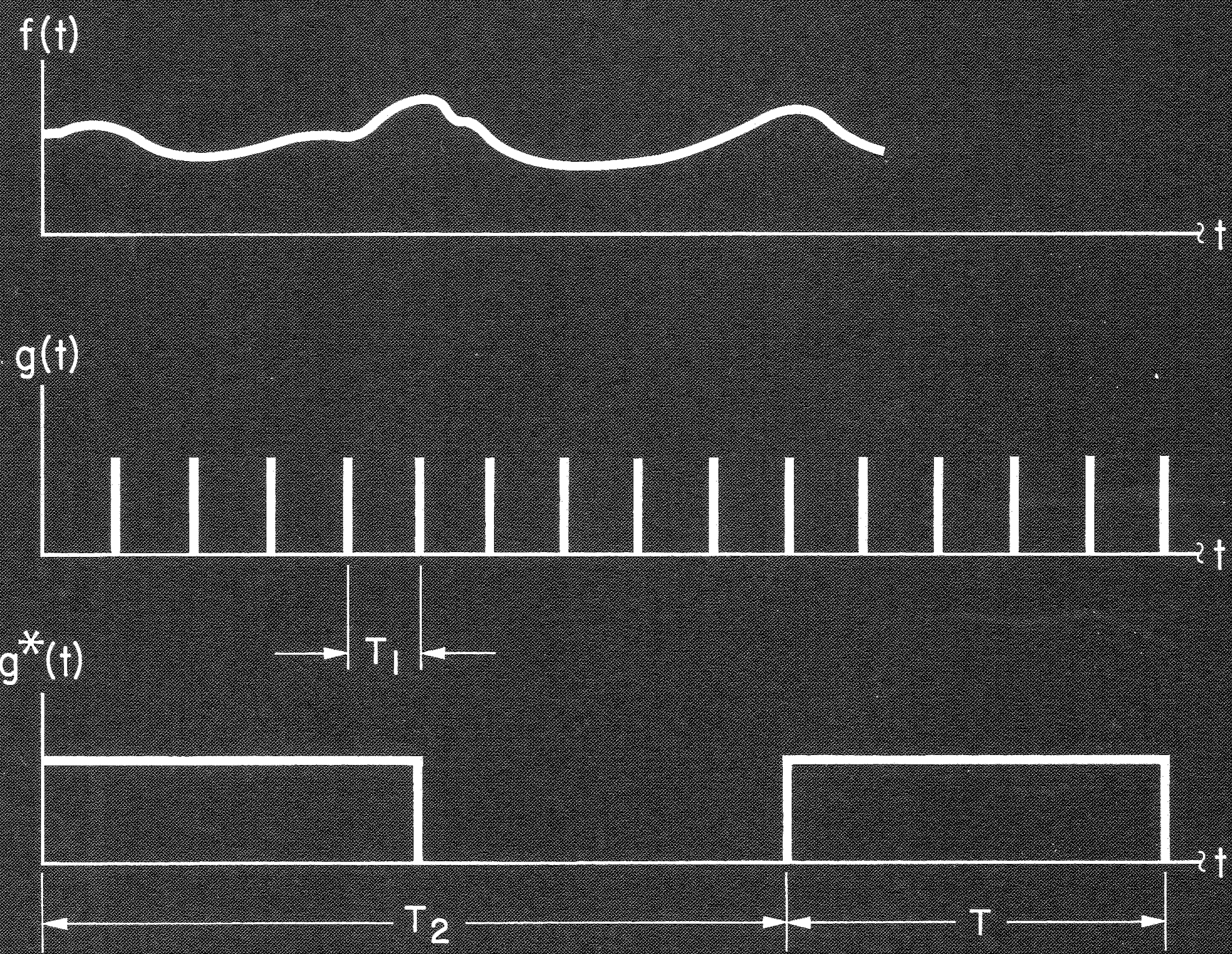

Figure 1

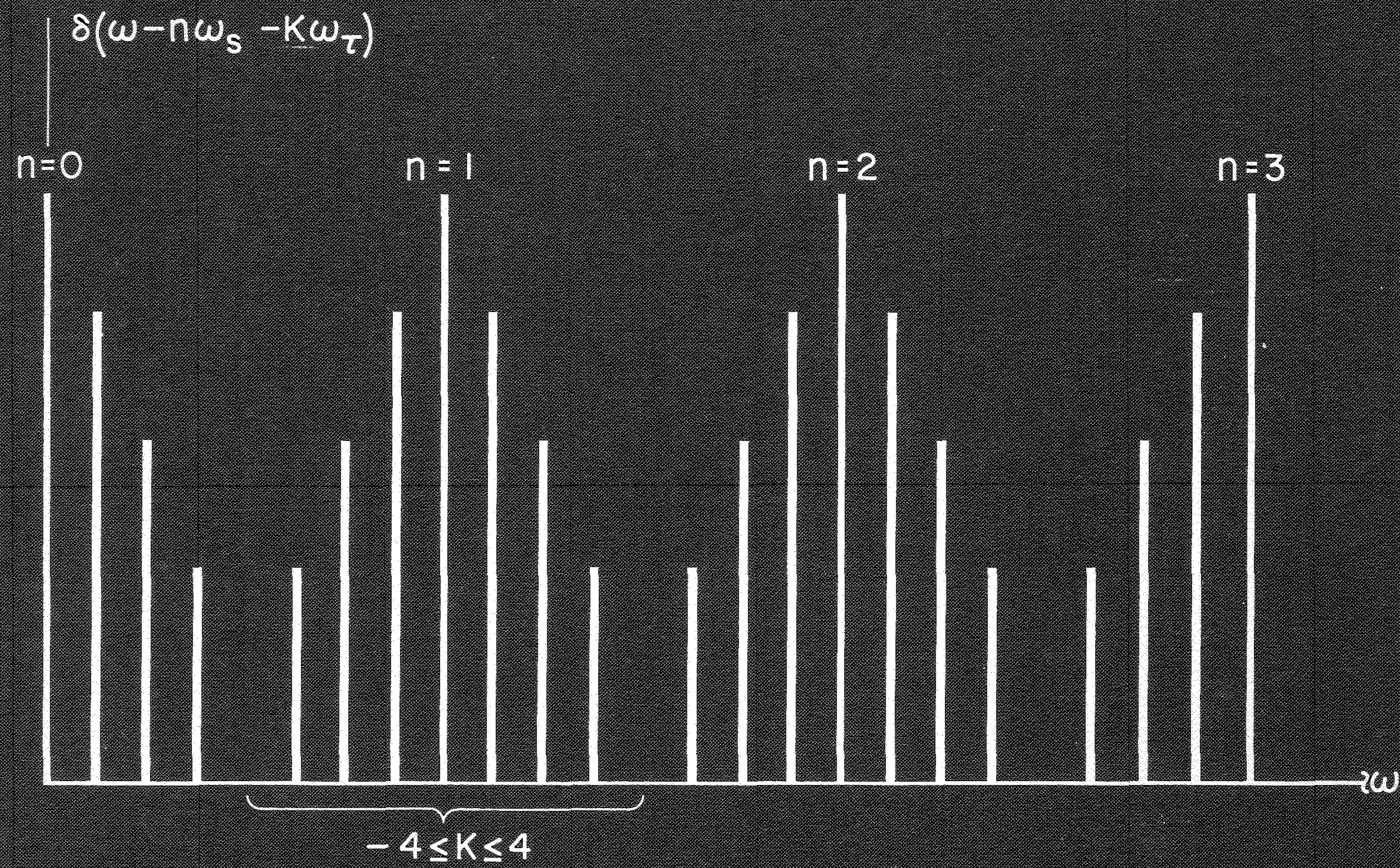

Figure 2

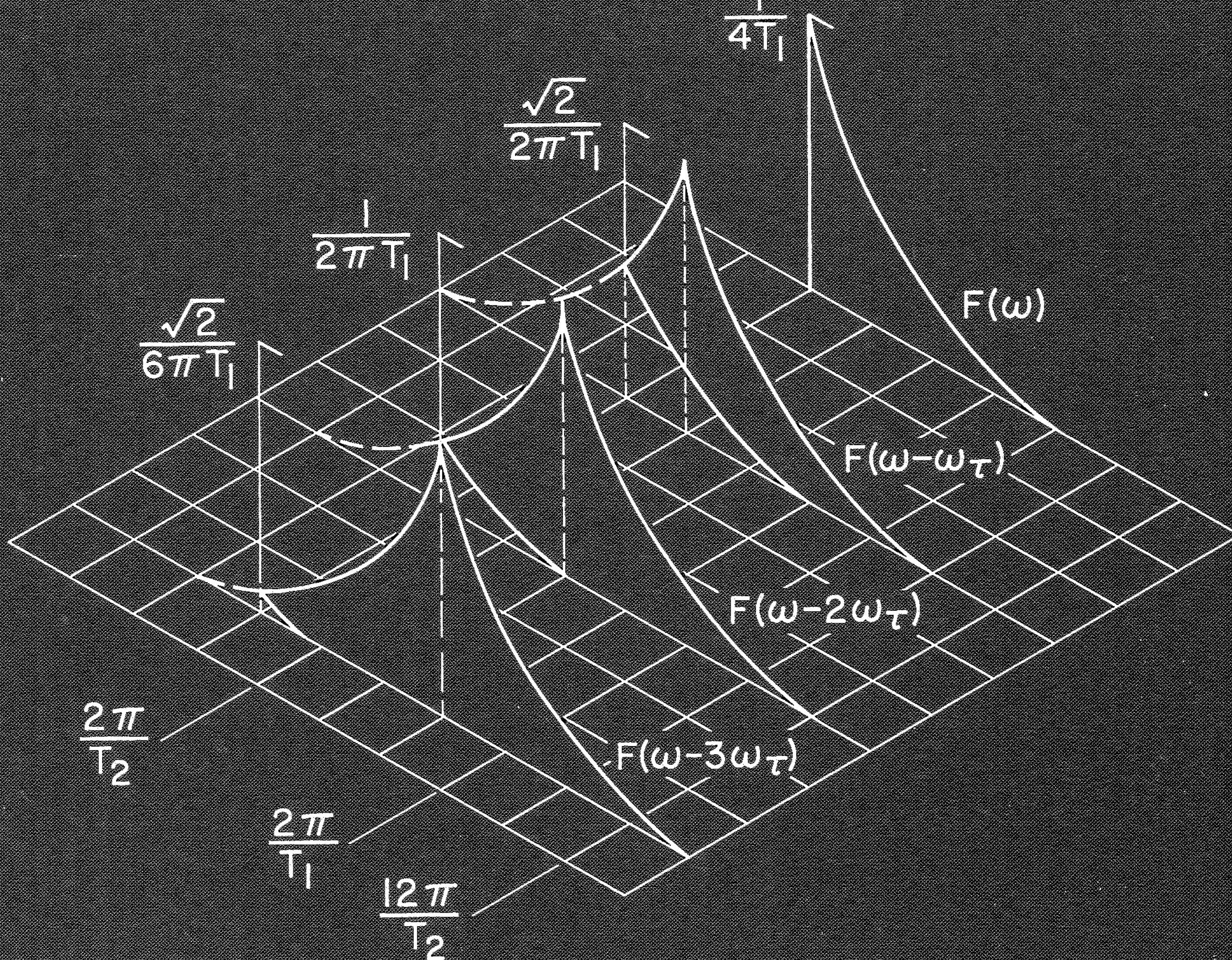

Figure 3

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
AMES RESEARCH CENTER, MOFFETT FIELD, CALIFORNIA

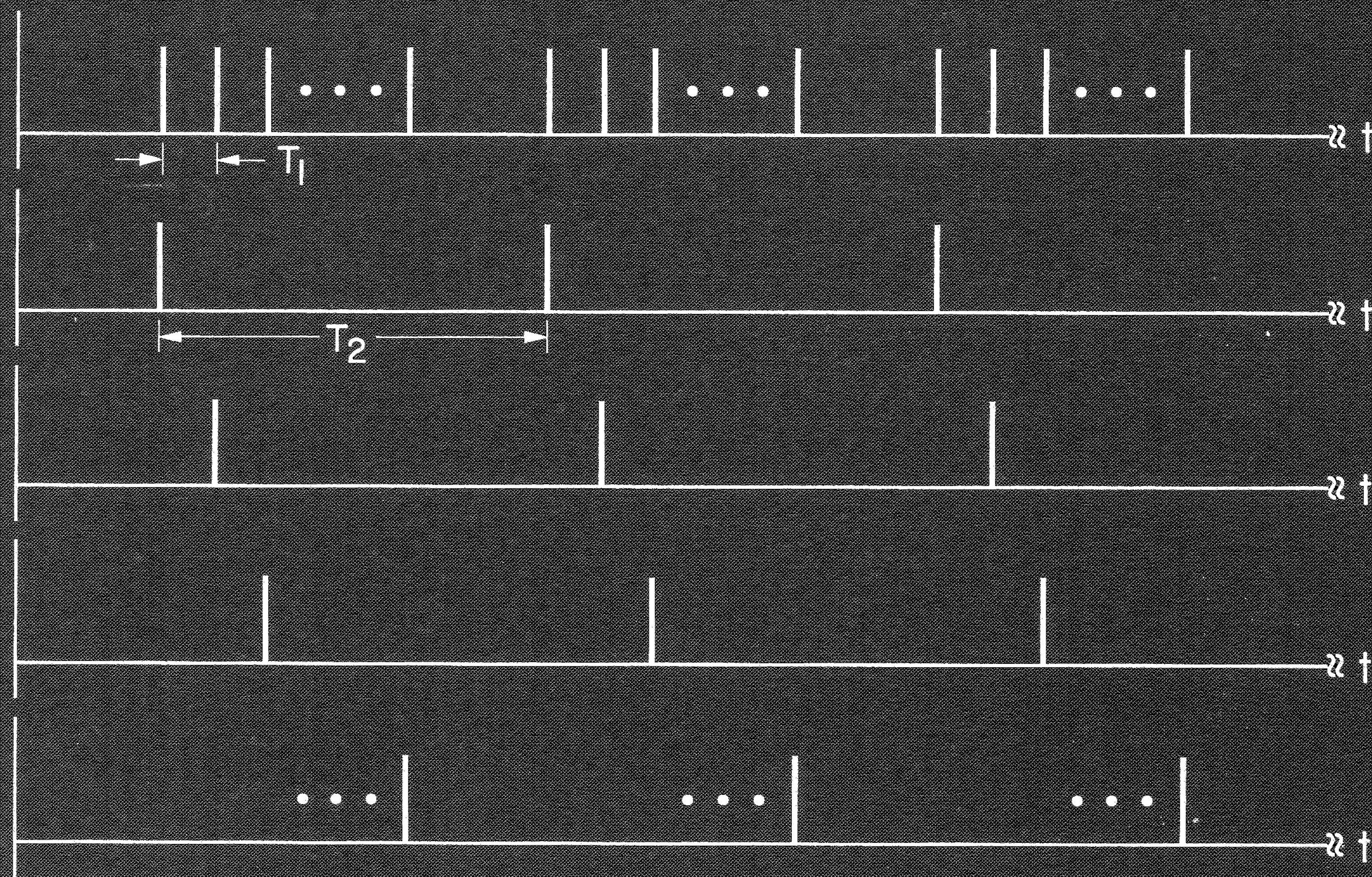

Figure 4

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
AMES RESEARCH CENTER, MOFFETT FIELD, CALIFORNIA